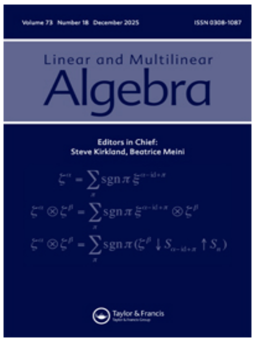

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Classification in a rotational flow of two-dimensional algebras

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ABSTRACT

In this paper, we examine a time-dependent family of two-dimensional algebras. We investigate the conditions under which any two algebras from this family, formed at different times, are isomorphic. Our findings reveal that the flow comprises of uncountable pairwise non-isomorphic algebras, including one commutative algebra. Additionally, we compare our results with a previously established classification of two-dimensional real algebras.

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1. Introduction

The paper [16] introduces a concept of time-dependent family of finite-dimensional algebras, called as a 'flow of algebras'. This flow can be seen as a continuous-time dynamical system, where the states are finite-dimensional algebras with cubic matrices of structural constants that satisfy Kolmogorov–Chapman equation (KCE) [16,17,28]. There are several kinds of multiplications between cubic matrices [2,20]. Therefore one has to fix a multiplication and then consider the KCE with respect to the fixed multiplication [17]. A flow of algebras is two-parametric family, which is a generalization of (one-parametric) deformation of algebras motivated in physics [15] and moduli theory [12].

In [6], Markov processes of cubic matrices are studied which are a two-parametric family of cubic stochastic matrices satisfying the KCE. In the book [26], several dynamical systems of biological models such as dynamics generated by Markov processes of cubic stochastic matrices; dynamics of sex-linked population; dynamical systems generated by a gonosomal evolution operator; dynamical system and an evolution algebra of mosquito

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population; and ocean ecosystems were studied. This book also gives motivations of such investigations.

In [5], the notion of chain of evolution algebras is introduced that is a particular case of the flow of algebras. Classification in chains of two-dimensional and three-dimensional evolution algebras are given in recent papers [21,22].

In this paper, we consider a flow of two-dimensional algebras found in [16]. We find condition (depending on time) for two algebras of this flow to be isomorphic at different times. Moreover, we show that the flow comprises of uncountable pairwise non-isomorphic algebras, including one commutative algebra. We compare our results with known classification of two-dimensional real algebras.

1.1. Cubic matrices

Following [2,18,20,25] recall the notion of cubic matrix and different associative multiplication rules of cubic matrices: a cubic matrix $Q=(q_{ijk})_{i,j,k=1}^{m}$ is an m^3-dimensional vector which can be uniquely written as

$$Q=\sum_{i,j,k=1}^{m} q_{ijk}E_{ijk},$$

where E_{ijk} denotes the cubic unit (basis) matrix, i.e. E_{ijk} is a m^3-cubic matrix whose (i,j,k)th entry is equal to 1 and all other entries are equal to 0.

Denote by $\mathfrak{C}$ the set of all cubic matrices over a field $\mathbb{F}$. Then $\mathfrak{C}$ is an m^3-dimensional vector space over $\mathbb{F}$, i.e. for any matrices $A=(a_{ijk})_{i,j,k=1}^{m}, B=(b_{ijk})_{i,j,k=1}^{m}\in\mathfrak{C},\ \lambda\in\mathbb{F}$, we have

$$A+B:=(a_{ijk}+b_{ijk})_{i,j,k=1}^{m}\in\mathfrak{C},\quad \lambda A:=(\lambda a_{ijk})_{i,j,k=1}^{m}\in\mathfrak{C}.$$

In [18] (see also [17]), some simple versions of multiplications between cubic matrices are given. Denote $I=\{1,2,\dots,m\}$. Following [20], we define the following multiplications for basis matrices E_{ijk}:

$$E_{ijk}*_a E_{lnr}=\delta_{kl}E_{ia(j,n)r}, \tag{1}$$

where $a: I\times I\to I,\ (j,n)\mapsto a(j,n)\in I$, is an arbitrary associative binary operation and δ_{kl} is the Kronecker symbol.

1.2. Two-dimensional algebras over $\mathbb{R}$

The classification problem of finite dimensional algebras is important in algebra. In [3], author considered such problem for two-dimensional algebras over the field of real numbers $\mathbb{R}$ and gave classifications of two-dimensional general, commutative, commutative Jordan, division and evolution real algebras. In [1], authors considered the problem over algebraically closed fields.

Now we use some notions provided in [3]. Let $\mathbb{A}$ be any two-dimensional algebra over a field $\mathbb{F}$ and $e=\{e_1,e_2\}$ be a basis for $\mathbb{A}$. If $e_i\cdot e_j=A^1_{i,j}e_1+A^2_{i,j}e_2,\ i,j=1,2$ is a multiplication table for e, then the matrix of structural constants of $\mathbb{A}$ with respect to the basis e is as follows:

$$\mathcal{A}=\begin{pmatrix} A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^1 & A_{2,2}^1 \\ A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^2 & A_{2,2}^2 \end{pmatrix}. \tag{2}$$

Let $\mathbb{F}=\mathbb{R}$ and for the simplicity, we use the notation

$$\mathcal{A}=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix},$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4$ stand for any elements of $\mathbb{R}$.

Theorem 1.1 ([3]): *Any non-trivial two-dimensional real algebra is isomorphic to only one of the following listed, by their matrices of structural constants, algebras:*

$$\mathcal{A}_1(\mathbf{c})=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}, \quad \text{where } \mathbf{c}=(\alpha_1,\alpha_2,\alpha_4,\beta_1)\in\mathbb{R}^4,$$

$$\mathcal{A}_2(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \beta_1\geq 0,\ \mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{R}^3,$$

$$\mathcal{A}_3(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \beta_1\geq 0,\ \mathbf{c}=(\alpha_1,\beta_1,\beta_2)\in\mathbb{R}^3,$$

$$\mathcal{A}_4(\mathbf{c})=\begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}, \quad \text{where } \mathbf{c}=(\beta_1,\beta_2)\in\mathbb{R}^2,$$

$$\mathcal{A}_5(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c}=(\alpha_1,\beta_2)\in\mathbb{R}^2,$$

$$\mathcal{A}_6(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c}=\alpha_1\in\mathbb{R},$$

$$\mathcal{A}_7(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \beta_1\geq 0, \mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{R}^2,$$

$$\mathcal{A}_8(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \beta_1\geq 0, \mathbf{c}=(\alpha_1,\beta_1)\in\mathbb{R}^2,$$

$$\mathcal{A}_9(\mathbf{c})=\begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}, \quad \text{where } \mathbf{c}=\beta_1\in\mathbb{R},$$

$$\mathcal{A}_{10}(\mathbf{c})=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \quad \text{where } \mathbf{c}=\alpha_1\in\mathbb{R},$$

$$\mathcal{A}_{11}=\begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}, \quad \mathcal{A}_{12}=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathcal{A}_{13}=\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}, \quad \mathcal{A}_{14}=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\mathcal{A}_{15}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

As shown in the following example, two algebras with different parameter values, belonging to the same class as defined in Theorem 1.1 may not be isomorphic.

Example 1.1: We consider the algebra $\mathcal{A}_1(\mathbf{c})$ with parameters $\mathbf{c}$ as follows: $\alpha_1=\alpha_2=\alpha_4=\beta_1=0$ and $\alpha_1=\alpha_2=\alpha_4=0$, $\beta_1=1$, then we have the following algebras:

$$\mathcal{A}_1(0,0,0,0)=\begin{pmatrix}0&0&1&0\\0&0&1&0\end{pmatrix},\quad \mathcal{A}_1(0,0,0,1)=\begin{pmatrix}0&0&1&0\\1&0&1&0\end{pmatrix}.$$

The multiplication tables for these algebras are as follows:

$$\mathcal{A}_1(0,0,0,0):\ e_1e_1=0,\quad e_1e_2=0,\quad e_2e_1=e_1+e_2,\quad e_2e_2=0;$$
$$\mathcal{A}_1(0,0,0,1):\ \widetilde{e}_1\widetilde{e}_1=\widetilde{e}_2,\quad \widetilde{e}_1\widetilde{e}_2=0,\quad \widetilde{e}_2\widetilde{e}_1=\widetilde{e}_1+\widetilde{e}_2,\quad \widetilde{e}_2\widetilde{e}_2=0.$$

Assume that $\mathcal{A}_1(0,0,0,0)$ and $\mathcal{A}_1(0,0,0,1)$ are isomorphic. Then there exists a change of basis as follows: $\begin{cases}\widetilde{e}_1=x_1e_1+x_2e_2\\ \widetilde{e}_2=y_1e_1+y_2e_2\end{cases}$ where $x_1,x_2,y_1,y_2\in\mathbb{R}$ and $x_1y_2\neq x_2y_1$. Hence,

$$\widetilde{e}_1\widetilde{e}_1=(x_1e_1+x_2e_2)(x_1e_1+x_2e_2)=x_2x_1e_2e_1=x_2x_1(e_1+e_2).$$

On the other hand, $\widetilde{e}_1\widetilde{e}_1=\widetilde{e}_2=y_1e_1+y_2e_2$. It follows that $y_1=y_2=x_2x_1$. Likewise, $\widetilde{e}_2\widetilde{e}_1=y_2x_1e_2e_1=y_2x_1(e_1+e_2)$ and $\widetilde{e}_2\widetilde{e}_1=\widetilde{e}_1+\widetilde{e}_2=(x_1+y_1)e_1+(x_2+y_2)e_2$. Then we have $x_1+y_1=x_2+y_2=y_2x_1$, it means that $x_1=x_2$. It is a contradiction to $x_1y_2\neq x_2y_1$.

Thus we have the following.

Remark 1.1: The parametric algebras in a given class of Theorem 1.1 may not be isomorphic for different parameters.

1.3. Flow of algebras

Following [16] we define a notion of flow of algebras (FA). Consider a family

$$\{A^{[s,t]}: s,t\in\mathbb{R},\ 0\leq s\leq t\}$$

of arbitrary m-dimensional algebras over the field $\mathbb{R}$, with basis $e_1,e_2,\dots,e_m$ and multiplication table

$$e_ie_j=\sum_{k=1}^m c_{ijk}^{[s,t]}e_k,\quad i,j=1,\dots,m. \tag{3}$$

Here parameters s, t are considered as time.

Denote by $\mathcal{M}^{[s,t]}=(c_{ijk}^{[s,t]})_{i,j,k=1,\dots m}$ the matrix of structural constants of $A^{[s,t]}$.

Definition 1.1: Fix an arbitrary multiplication of cubic matrices, say $*_\mu$.

A family $\{A^{[s,t]}: s,t\in\mathbb{R},\ 0\leq s\leq t\}$ of m-dimensional algebras over the field $\mathbb{R}$ is called a flow of algebras (FA) of type μ if the matrices $\mathcal{M}^{[s,t]}$ of structural constants satisfy the Kolmogorov–Chapman equation (for cubic matrices):

$$\mathcal{M}^{[s,t]}=\mathcal{M}^{[s,\tau]}*_\mu\mathcal{M}^{[\tau,t]},\quad \text{for all } 0\leq s<\tau<t. \tag{4}$$

Definition 1.2: An FA is called a time-homogeneous FA if the matrix $\mathcal{M}^{[s,t]}$ depends only on $t-s$. In this case, we write $\mathcal{M}^{[t-s]}$.

Definition 1.3: An FA is called periodic if its matrix $\mathcal{M}^{[s,t]}$ is periodic with respect to at least one of the variables s, t, i.e. (periodicity with respect to t) $\mathcal{M}^{[s,t+P]} = \mathcal{M}^{[s,t]}$ for all values of t. The constant P is called the period and is required to be nonzero.

Remark 1.2: Following [16], we give a comparison of FA with Deformations of Algebras: In the case of finite-dimensional algebras, a deformation refers to the transformation of a given algebra into another algebra with the same dimension [10]. This is equivalent to changing the multiplication table (structural constants) of the given algebra to another one. Often, the algebra at $t = 0$ is considered as the initial algebra, and the algebra at the current time t is considered as the current algebra. In [10] (see also [9]) deformations of an algebra A are given by a bilinear function f_t. That is, one considers the algebra A_t with multiplication f_t as the generic element of a 'one-parameter family of deformations of A'. Thus deformations of an algebra are given by the rule f_t, which has an explicit form. Similarly, a flow of algebras is given by $\mathcal{M}^{[s,t]}$ with the rule (4). Some relations of an FA can be also seen in the following example: a plane deformation [23] is restricted to the plane described by the basis vectors e_1, e_2. It is known that the deformation gradient of this plane deformation has the form

$$\mathbf{F} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where θ is the angle of rotation and λ_1, λ_2 are the principal stretches. Comparing this matrix with matrix (8) (see below) one can see that the corresponding FA and the plane deformation have similar matrices.

To construct an FA of type μ, one has to solve (4). In this paper, we consider one its solution for the case $m = 2$ and a fixed multiplication of cubic matrices called type C.

2. Two-dimensional flow of algebras

Let $m = 2$ and consider two-dimensional flow of algebras [16], which is defined by multiplication:

$$E_{ijk} \cdot E_{lnr} = \delta_{kl}\delta_{jn}E_{ijr}, \tag{5}$$

where δ_{ij} is the Kronecker symbol.

The multiplication (5) and corresponding FA are called type C in [16]. Extending this multiplication to arbitrary cubic matrices

$$A = (a_{ijk})_{i,j,k=1}^{m}, \quad B = (b_{ijk})_{i,j,k=1}^{m}, \quad C = (c_{ijk})_{i,j,k=1}^{m},$$

we get that the entries of $C = AB$ can be written as

$$c_{ijr} = \sum_{k=1}^{m} a_{ijk}b_{kjr}.$$

Write cubic matrix $\mathcal{M}^{[s,t]}$ for $m = 2$ in the following convenient form:

$$\mathcal{M}^{[s,t]} = \left(\begin{array}{cc|cc} c_{111}^{[s,t]} & c_{112}^{[s,t]} & c_{211}^{[s,t]} & c_{212}^{[s,t]} \\ c_{121}^{[s,t]} & c_{122}^{[s,t]} & c_{221}^{[s,t]} & c_{222}^{[s,t]} \end{array}\right). \tag{6}$$

In this case, Equation (4) has the following form:

$$c_{ijr}^{[s,t]} = c_{ij1}^{[s,\tau]} c_{1jr}^{[\tau,t]} + c_{ij2}^{[s,\tau]} c_{2jr}^{[\tau,t]}, \quad i, j, r = 1, 2.$$

This is a quadratic system of eight equations with eight unknown functions $c_{ijr}^{[s,t]}$ of two variables $s, t,\ 0 \le s < t$. If we consider four equations with $j = 1$, the unknowns in them do not participate in the other four equations with $j = 2$. Therefore the equations for $j = 1$ and $j = 2$ are independent. Hence it suffices to solve the system only for $j = 1$. Denote $a_{ir}^{[s,t]} = c_{i1r}^{[s,t]}$ and we have the following system:

$$\begin{cases} a_{11}^{[s,t]} = a_{11}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{21}^{[\tau,t]} \\ a_{12}^{[s,t]} = a_{11}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{22}^{[\tau,t]} \\ a_{21}^{[s,t]} = a_{21}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{21}^{[\tau,t]} \\ a_{22}^{[s,t]} = a_{21}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{22}^{[\tau,t]}. \end{cases} \tag{7}$$

The full set of solutions to the system (7) is not known yet. But there is a very wide class of its solutions, see [5,13,24,27]. One of these known solutions is the following matrix:

$$\begin{pmatrix} a_{11}^{[s,t]} & a_{12}^{[s,t]} \\ a_{21}^{[s,t]} & a_{22}^{[s,t]} \end{pmatrix} = \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}. \tag{8}$$

Remark 2.1: Note that the matrix (7,8) defines a linear rotation flow [19,23].

By $(\overline{a}_{ij}^{[s,t]})$ we denote the matrix transposed to matrix $(a_{ij}^{[s,t]})$. Any pair $(a_{ij}^{[s,t]})$, $(\overline{a}_{ij}^{[s,t]})$ of solutions of the system (7) generates an FA of type C corresponding to the matrix $\mathcal{M}^{[s,t]}$ with entries

$$c_{i1r}^{[s,t]} = a_{ir}^{[s,t]}, \quad c_{i2r}^{[s,t]} = \overline{a}_{ir}^{[s,t]}.$$

For the matrix (8) we get the following cubic matrix, which generates an FA of type C:

$$\mathcal{M}^{[s,t]} = \left(\begin{array}{cc|cc} \cos(t-s) & \sin(t-s) & -\sin(t-s) & \cos(t-s) \\ \cos(t-s) & -\sin(t-s) & \sin(t-s) & \cos(t-s) \end{array}\right). \tag{9}$$

We note that a cubic matrix $(c_{ijk})_{i,j,k=1}^m$ generates a commutative algebra iff $c_{ijk} = c_{jik}$. Hence the algebra $A^{[s,t]}$ of the FA corresponding to the cubic matrix (9) is commutative iff $\cos(t - s) = -\sin(t - s)$, i.e. $t = s + \frac{3\pi}{4} + \pi n$, $n = 0, 1, 2, \ldots$. Therefore this FA is almost non-commutative (with respect to Lebesgue measure on $\mathcal{T} = \{(s,t) : 0 \le s \le t\}$).

Note that this FA is a time-homogeneous, therefore we can consider it with respect to one time parameter $\widehat{t} = t - s$. So for convenience, we write $A^{[t]}$ instead of $A^{[s,t]}$.

Remark 2.2: The FA (9) is periodic and its period is 2π.

The main problem of this paper is to classify algebras of $A^{[t]}$, i.e. at different values of time FA forms various algebras, we need to find the time-dependent conditions under which algebras are isomorphic.

Let $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ are bases of algebras $A^{[t_1]}$ and $A^{[t_2]}$ respectively. We consider the following multiplication table of basis $\{e_1, e_2\}$ corresponding to (9):

$$\begin{aligned}
e_1e_1 &= c_{111}e_1 + c_{112}e_2 = \cos t_1 e_1 + \sin t_1 e_2,\\
e_1e_2 &= c_{121}e_1 + c_{122}e_2 = \cos t_1 e_1 - \sin t_1 e_2,\\
e_2e_1 &= c_{211}e_1 + c_{212}e_2 = -\sin t_1 e_1 + \cos t_1 e_2,\\
e_2e_2 &= c_{221}e_1 + c_{222}e_2 = \sin t_1 e_1 + \cos t_1 e_2.
\end{aligned} \tag{10}$$

Assume that $A^{[t_1]}$ and $A^{[t_2]}$ are isomorphic, i.e. we assume the isomorphism map e_i to e'_i. Then there exists a change of basis as follows:

$$\begin{cases} e'_1 = x_1e_1 + x_2e_2\\ e'_2 = y_1e_1 + y_2e_2, \end{cases} \tag{11}$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $x_1y_2 \neq x_2y_1$.

We can also write a multiplication table similar to (10) for $\{e'_1, e'_2\}$. On other hand from (11), we have

$$\begin{aligned}
e'_1e'_1 &= x_1^2e_1e_1 + x_1x_2e_1e_2 + x_2x_1e_2e_1 + x_2^2e_2e_2,\\
e'_1e'_2 &= x_1y_1e_1e_1 + x_1y_2e_1e_2 + x_2y_1e_2e_1 + x_2y_2e_2e_2,\\
e'_2e'_1 &= x_1y_1e_1e_1 + x_2y_1e_1e_2 + x_1y_2e_2e_1 + x_2y_2e_2e_2,\\
e'_2e'_2 &= y_1^2e_1e_1 + y_1y_2e_1e_2 + y_2y_1e_2e_1 + y_2^2e_2e_2.
\end{aligned}$$

If we use (10), then after some easy calculations we get

$$\begin{aligned}
e'_1e'_1 = {} & (x_1^2 \cos t_1 + x_1x_2(\cos t_1 - \sin t_1) + x_2^2 \sin t_1)e_1+\\
& + (x_1^2 \sin t_1 + x_1x_2(\cos t_1 - \sin t_1) + x_2^2 \cos t_1)e_2,\\
e'_1e'_2 = {} & (x_1y_1 \cos t_1 + x_1y_2 \cos t_1 - x_2y_1 \sin t_1 + x_2y_2 \sin t_1)e_1+\\
& + (x_1y_1 \sin t_1 - x_1y_2 \sin t_1 + x_2y_1 \cos t_1 + x_2y_2 \cos t_1)e_2,\\
e'_2e'_1 = {} & (x_1y_1 \cos t_1 + x_2y_1 \cos t_1 - x_1y_2 \sin t_1 + x_2y_2 \sin t_1)e_1+\\
& + (x_1y_1 \sin t_1 - x_2y_1 \sin t_1 + x_1y_2 \cos t_1 + x_2y_2 \cos t_1)e_2,\\
e'_2e'_2 = {} & (y_1^2 \cos t_1 + y_1y_2(\cos t_1 - \sin t_1) + y_2^2 \sin t_1)e_1+\\
& + (y_1^2 \sin t_1 + y_1y_2(\cos t_1 - \sin t_1) + y_2^2 \cos t_1)e_2.
\end{aligned} \tag{12}$$

If we use the multiplication table of $\{e'_1, e'_2\}$ and the change (11), then we have

$$\begin{aligned}
e'_1e'_1 &= (x_1 \cos t_2 + y_1 \sin t_2)e_1 + (x_2 \cos t_2 + y_2 \sin t_2)e_2,\\
e'_1e'_2 &= (x_1 \cos t_2 - y_1 \sin t_2)e_1 + (x_2 \cos t_2 - y_2 \sin t_2)e_2,\\
e'_2e'_1 &= (-x_1 \sin t_2 + y_1 \cos t_2)e_1 + (-x_2 \sin t_2 + y_2 \cos t_2)e_2,\\
e'_2e'_2 &= (x_1 \sin t_2 + y_1 \cos t_2)e_1 + (x_2 \sin t_2 + y_2 \cos t_2)e_2.
\end{aligned} \tag{13}$$

By equating the coefficients of the corresponding terms in systems (12) and (13), we obtain the following system:

$$\begin{cases} x_1^2\cos t_1 + x_1x_2(\cos t_1 - \sin t_1) + x_2^2\sin t_1 = x_1\cos t_2 + y_1\sin t_2 \\ x_1^2\sin t_1 + x_1x_2(\cos t_1 - \sin t_1) + x_2^2\cos t_1 = x_2\cos t_2 + y_2\sin t_2 \\ x_1y_1\cos t_1 + x_1y_2\cos t_1 - x_2y_1\sin t_1 + x_2y_2\sin t_1 = x_1\cos t_2 - y_1\sin t_2 \\ x_1y_1\sin t_1 - x_1y_2\sin t_1 + x_2y_1\cos t_1 + x_2y_2\cos t_1 = x_2\cos t_2 - y_2\sin t_2 \\ x_1y_1\cos t_1 + x_2y_1\cos t_1 - x_1y_2\sin t_1 + x_2y_2\sin t_1 = -x_1\sin t_2 + y_1\cos t_2 \\ x_1y_1\sin t_1 - x_2y_1\sin t_1 + x_1y_2\cos t_1 + x_2y_2\cos t_1 = -x_2\sin t_2 + y_2\cos t_2 \\ y_1^2\cos t_1 + y_1y_2(\cos t_1 - \sin t_1) + y_2^2\sin t_1 = x_1\sin t_2 + y_1\cos t_2 \\ y_1^2\sin t_1 + y_1y_2(\cos t_1 - \sin t_1) + y_2^2\cos t_1 = x_2\sin t_2 + y_2\cos t_2 \\ x_1y_2 \neq x_2y_1. \end{cases}$$

We write this system in the following convenient form:

$$\begin{cases} (x_1^2 + x_1x_2)\cos t_1 + (x_2^2 - x_1x_2)\sin t_1 = x_1\cos t_2 + y_1\sin t_2 \\ (x_2^2 + x_1x_2)\cos t_1 + (x_1^2 - x_1x_2)\sin t_1 = x_2\cos t_2 + y_2\sin t_2 \\ (x_1y_1 + x_1y_2)\cos t_1 + (x_2y_2 - x_2y_1)\sin t_1 = x_1\cos t_2 - y_1\sin t_2 \\ (x_2y_1 + x_2y_2)\cos t_1 + (x_1y_1 - x_1y_2)\sin t_1 = x_2\cos t_2 - y_2\sin t_2 \\ (x_1y_1 + x_2y_1)\cos t_1 + (x_2y_2 - x_1y_2)\sin t_1 = y_1\cos t_2 - x_1\sin t_2 \\ (x_1y_2 + x_2y_2)\cos t_1 + (x_1y_1 - x_2y_1)\sin t_1 = y_2\cos t_2 - x_2\sin t_2 \\ (y_1^2 + y_1y_2)\cos t_1 + (y_2^2 - y_1y_2)\sin t_1 = y_1\cos t_2 + x_1\sin t_2 \\ (y_2^2 + y_1y_2)\cos t_1 + (y_1^2 - y_1y_2)\sin t_1 = y_2\cos t_2 + x_2\sin t_2 \\ x_1y_2 \neq x_2y_1. \end{cases} \tag{14}$$

For convenience, we denote

$$x_1 + x_2 = u, \quad y_1 + y_2 = v, \quad x_1 - x_2 = \alpha, \quad y_1 - y_2 = \beta. \tag{15}$$

Then the system (14) will be in the following form:

$$\begin{cases} x_1u\cos t_1 - x_2\alpha\sin t_1 = x_1\cos t_2 + y_1\sin t_2 \\ x_2u\cos t_1 + x_1\alpha\sin t_1 = x_2\cos t_2 + y_2\sin t_2 \\ x_1v\cos t_1 - x_2\beta\sin t_1 = x_1\cos t_2 - y_1\sin t_2 \\ x_2v\cos t_1 + x_1\beta\sin t_1 = x_2\cos t_2 - y_2\sin t_2 \\ y_1u\cos t_1 - y_2\alpha\sin t_1 = y_1\cos t_2 - x_1\sin t_2 \\ y_2u\cos t_1 + y_1\alpha\sin t_1 = y_2\cos t_2 - x_2\sin t_2 \\ y_1v\cos t_1 - y_2\beta\sin t_1 = y_1\cos t_2 + x_1\sin t_2 \\ y_2v\cos t_1 + y_1\beta\sin t_1 = y_2\cos t_2 + x_2\sin t_2 \\ x_1y_2 \neq x_2y_1. \end{cases} \tag{16}$$

Note that $u^2 + v^2 \neq 0$ and $\alpha^2 + \beta^2 \neq 0$, otherwise $x_1y_2 = x_2y_1$. From (15), we have

$$x_1 = \frac{u+\alpha}{2}, \quad x_2 = \frac{u-\alpha}{2}, \quad y_1 = \frac{v+\beta}{2}, \quad y_2 = \frac{v-\beta}{2}.$$

Then the expression $x_1y_2 \neq x_2y_1$ takes the form $\alpha v \neq u\beta$ and by adding each of two equations of system (16), we obtain

$$\begin{cases} u^2\cos t_1 + \alpha^2 \sin t_1 = u\cos t_2 + v\sin t_2 \\ uv\cos t_1 + \alpha\beta \sin t_1 = u\cos t_2 - v\sin t_2 \\ uv\cos t_1 + \alpha\beta \sin t_1 = v\cos t_2 - u\sin t_2 \\ v^2\cos t_1 + \beta^2 \sin t_1 = v\cos t_2 + u\sin t_2 \\ \alpha v \neq u\beta. \end{cases} \tag{17}$$

If the system (14) has a solution $\{x_1, x_2, y_1, y_2\}$, then the system (17) also has a solution $\{u, v, \alpha, \beta\}$. By our assumption, the system (14) has a non-trivial solution, then the system (17) also has a non-trivial solution.

From the second and third equations of the system (17), we get

$$u\cos t_2 - v\sin t_2 = v\cos t_2 - u\sin t_2,$$
$$u(\cos t_2 + \sin t_2) - v(\cos t_2 + \sin t_2) = 0.$$

Therefore

$$(u - v)(\cos t_2 + \sin t_2) = 0 \tag{18}$$

From the last equality, we have two cases.

Case 1. Let $u = v$ then $x_1 + x_2 = y_1 + y_2 = u$.

In this case, the system (17) will be in the following form:

$$\begin{cases} u^2\cos t_1 + \alpha^2 \sin t_1 = u\cos t_2 + u\sin t_2 \\ u^2\cos t_1 + \alpha\beta \sin t_1 = u\cos t_2 - u\sin t_2 \\ u^2\cos t_1 + \beta^2 \sin t_1 = u\cos t_2 + u\sin t_2 \\ \alpha v \neq u\beta. \end{cases} \tag{19}$$

Note that $x_i \neq y_i$, $i = 1, 2$ otherwise $x_1y_2 = x_2y_1$. From the first and third equations of this system, we get

$$\alpha^2 \sin t_1 = \beta^2 \sin t_1. \tag{20}$$

Case 1.1. First we consider the case $\sin t_1 = 0$ then $\cos t_1 = \pm 1$.

Let $\begin{cases} \sin t_1 = 0 \\ \cos t_1 = 1 \end{cases}$, then from (19) we have $\begin{cases} u^2 = u\cos t_2 + u\sin t_2 \\ u^2 = u\cos t_2 - u\sin t_2 \end{cases}$.

We know that $u \neq 0$, then

$$\begin{cases} u = \cos t_2 + \sin t_2 \\ u = \cos t_2 - \sin t_2 \end{cases}$$

From this system we get $\sin t_2 = 0$, then $\cos t_2 = \pm 1$. In this case $u = \pm 1$.Therefore $\begin{cases} x_1 + x_2 = \pm 1 \\ y_1 + y_2 = \pm 1 \end{cases}$, solutions of this systems are

$$x_1 = \gamma, \quad x_2 = \pm 1 - \gamma, \quad y_1 = \mu, \quad y_2 = \pm 1 - \mu, \quad \gamma \neq \mu \quad \gamma, \mu \in \mathbb{R}. \tag{21}$$

The case $\begin{cases} \sin t_1 = 0 \\ \cos t_1 = -1 \end{cases}$ is similar to the above case.

In this case from $\sin t_1 = 0$ it follows that $\sin t_2 = 0$ and solutions of the system (14) are (21). And we have the following isomorphic two-dimensional algebras A_1 and A_{-1} with structural constants matrix

$$\mathcal{M}_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right) \quad \text{and} \quad \mathcal{M}_{-1} = \left(\begin{array}{cc|cc} -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \end{array}\right) \quad \text{respectively.}$$

Indeed if we take the change of basis $e_i' = -e_i,\ i = 1, 2$, then it is easy to see that they are isomorphic.

Case 1.2. Now we consider $\sin t_1 \neq 0$. From the equality (20), we have $\alpha^2 = \beta^2$. It means that $\alpha = \pm\beta$ and note that $\beta \neq 0$.

Case 1.2.1. If $\alpha = \beta$, then $\alpha v = u\beta$ (since we are considering the case $u = v$). It is contradiction.

Case 1.2.2. Let $\alpha = -\beta$, then we have the following system:

$$\begin{cases} x_1 - x_2 = \alpha \\ x_1 + x_2 = u \\ y_1 - y_2 = -\alpha \\ y_1 + y_2 = u \end{cases}$$

Solution of this system is $x_1 = y_2 = \frac{u+\alpha}{2}$, $x_2 = y_1 = \frac{u-\alpha}{2}$. In this case

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

and the system (19) will be

$$\begin{cases} u^2 \cos t_1 + \alpha^2 \sin t_1 = u \cos t_2 + u \sin t_2 \\ u^2 \cos t_1 - \alpha^2 \sin t_1 = u \cos t_2 - u \sin t_2 \\ \alpha v \neq u\beta. \end{cases} \tag{22}$$

By adding the first two equations of this system, we have

$$u \cos t_1 = \cos t_2. \tag{23}$$

Case 1.2.2.1. Let $\cos t_1 = 0$, then $\cos t_2 = 0$. Since $\alpha = -\beta$, in this case the system (16) will be

$$\begin{cases} -x_2\alpha \sin t_1 = y_1 \sin t_2 \\ x_1\alpha \sin t_1 = y_2 \sin t_2 \\ y_2\alpha \sin t_1 = x_1 \sin t_2 \\ y_1\alpha \sin t_1 = -x_2 \sin t_2 \\ x_1y_2 \neq x_2y_1. \end{cases} \tag{24}$$

Assume that $x_1 \neq 0$. Since we are considering the case $u = v$ and $\alpha = -\beta$, we get from (15) that $x_1 = y_2$, $x_2 = y_1$. Then we have from (24) $\alpha = \frac{\sin t_2}{\sin t_1}$, $x_2 = y_1 = 0$ and in this case $\alpha = \frac{\sin t_2}{\sin t_1} = \pm 1$. From (22) $u = \alpha = \pm 1$, it means that $x_1 = \pm 1$.

In this case from $\cos t_1 = 0$ it follows that $\cos t_2 = 0$ and solutions of the system (14) are $x_1 = y_2 = \pm 1, x_2 = y_1 = 0$. And we have the following isomorphic two-dimensional algebras A_0^+ and A_0^- with structural constants matrix

$$\mathcal{M}_0^+ = \left(\begin{array}{cc|cc} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \quad \text{and} \quad \mathcal{M}_0^- = \left(\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

respectively.

Proposition 2.1: *A_0^+ is not isomorphic to A_1.*

Proof: Indeed if we assume that they are isomorphic, then we have a system similar to (14). And this system will be in the following form:

$$\begin{cases} x_1^2 + x_1x_2 = y_1 \\ x_2^2 + x_1x_2 = y_2 \\ x_1y_1 + x_1y_2 = -y_1 \\ x_2y_1 + x_2y_2 = -y_2 \\ x_1y_1 + x_2y_1 = -x_1 \\ x_1y_2 + x_2y_2 = -x_2 \\ y_1^2 + y_1y_2 = x_1 \\ y_2^2 + y_1y_2 = x_2 \\ x_1y_2 \neq x_2y_1. \end{cases} \tag{25}$$

If we add the equations of the system, then we obtain

$$(x_1 + x_2 + y_1 + y_2)^2 = 0. \tag{26}$$

We get the following system by adding the first and last two equations:

$$\begin{cases} (x_1 + x_2)^2 = y_1 + y_2 \\ (y_1 + y_2)^2 = x_1 + x_2. \end{cases}$$

In turn, we form

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 = y_1 + y_2 + x_1 + x_2$$

from these equations. From the last equation and (26) we have $x_1 + x_2 = y_1 + y_2 = 0$. It is contradiction to $x_1y_2 \neq x_2y_1$. ■

Case 1.2.2.2. Let $\cos t_1 \neq 0$, then $\cos t_2 \neq 0$ (see (23)). From (23), we obtain

$$u = \frac{\cos t_2}{\cos t_1}. \tag{27}$$

If we use the equality (27) in the system (16), then we have the system (24) again (note that $u = v$ and $\alpha = -\beta$).

In this case $\sin t_2 \neq 0$, otherwise we get $x_1 = x_2 = y_1 = y_2 = 0$ from (24) and $\sin t_1 \neq 0$, $\alpha \neq 0$ (see Case 1.2.), it is contradiction.

If we assume $x_1 \neq 0$, then we have $\alpha = \frac{\sin t_2}{\sin t_1}$ and $x_2 = y_1 = 0$. Indeed, in the case we are studying, $\alpha = -\beta$ (see Case 1.2.2.) and $x_1 = y_2 = \frac{u+\alpha}{2}$, $x_2 = y_1 = \frac{u-\alpha}{2}$. Applying the last equalities to (24), we get $\alpha = \frac{\sin t_2}{\sin t_1}$ and $x_2 = y_1 = 0$. It implies that $x_1 = u = \alpha = y_2$ (see (15)).

In this case by system (16) we have that the system (14) also has a unique solution $x_1 = y_2 = \alpha = u = \frac{\cos t_2}{\cos t_1} \neq 0$, $x_2 = y_1 = 0$. Since $\alpha = u$, the equality

$$\frac{\sin t_2}{\sin t_1} = \frac{\cos t_2}{\cos t_1}$$

must be satisfied for the system (14) to have a solution. Consequently $\sin(t_2 - t_1) = 0$, i.e. $t_2 = t_1 + \pi k,\ k \in Z$.

If $x_1 = 0$, then we can assume $x_2 \neq 0$. In this case, $x_1 = y_2 = 0, x_2 = y_1 = u = -\alpha \neq 0$ (see (15)) and $\alpha = -\frac{\sin t_2}{\sin t_1}$ (see (24)), also $\sin(t_2 - t_1) = 0$.

So, if $\sin t_1 \sin t_2 \cos t_1 \cos t_2 \neq 0$, then the system (14) has a solution

$x_1 = y_2 = \frac{\cos t_2}{\cos t_1}, x_2 = y_1 = 0$ or $x_1 = y_2 = 0, x_2 = y_1 = \frac{\cos t_2}{\cos t_1}$, where $t_2 = t_1 + \pi k,\ k \in Z$.

In this case, we have the following two-dimensional algebras $A^+_{\cos t}$ and $A^-_{\cos t}$ with structural constants matrix

$$\mathcal{M}^+_{\cos t} = \left(\begin{array}{cc|cc} \cos t & \sqrt{1-\cos^2 t} & -\sqrt{1-\cos^2 t} & \cos t \\ \cos t & -\sqrt{1-\cos^2 t} & \sqrt{1-\cos^2 t} & \cos t \end{array}\right) \quad \text{and}$$

$$\mathcal{M}^-_{\cos t} = \left(\begin{array}{cc|cc} \cos t & -\sqrt{1-\cos^2 t} & \sqrt{1-\cos^2 t} & \cos t \\ \cos t & \sqrt{1-\cos^2 t} & -\sqrt{1-\cos^2 t} & \cos t \end{array}\right) \quad \text{respectively,}$$

where $\cos t \in (-1;0) \cup (0;1)$ and the sign above $\mathcal{M}_{\cos t}$ represents the sign of the $\sin t$ in the matrix (9).

The following theorem answers the question of whether the algebras found are different algebras.

Theorem 2.1: (1) $A^+_{\cos t}$ *is isomorphic to* $A^-_{-\cos t}$ *for any* $\cos t \in (-1;1) \setminus \{0\}$.
(2) $A^+_{\cos t_1}$ *is not isomorphic to* $A^+_{\cos t_2}$ *for any* $\cos t_1, \cos t_2 \in (0;1), \cos t_1 \neq \cos t_2$.
(3) $A^-_{\cos t_1}$ *is not isomorphic to* $A^-_{\cos t_2}$ *for any* $\cos t_1, \cos t_2 \in (0;1), \cos t_1 \neq \cos t_2$.
(4) $A^+_{\cos t_1}$ *is not isomorphic to* $A^-_{\cos t_2}$ *for any* $\cos t_1, \cos t_2 \in (0;1)$.

Proof: (1) Let $\{e_1, e_2\}$ and $\{e_1^*, e_2^*\}$ be bases of $A^+_{\cos t}$ and $A^-_{-\cos t}$ respectively. If we take the change $e_1^* = -e_1, e_2^* = -e_2$ of basis then it is easy to see that they are isomorphic.

(2) Assume that $A^+_{\cos t_1}$ is isomorphic to $A^+_{\cos t_2}$ for some $\cos t_1, \cos t_2 \in (0;1), \cos t_1 \neq \cos t_2$. If $\{e_1, e_2\}$, $\{e_1^*, e_2^*\}$ be bases of $A^+_{\cos t_1}$ and $A^+_{\cos t_2}$ respectively, then from the above-mentioned results (see Case 1.2.2.2) it is easy to see that there are only the following changes

of basis:

$$(a)\ \begin{cases} e_1^* = e_1 \\ e_2^* = e_2 \end{cases},\quad (b)\ \begin{cases} e_1^* = -e_1 \\ e_2^* = -e_2 \end{cases},\quad (c)\ \begin{cases} e_1^* = e_2 \\ e_2^* = e_1 \end{cases},\quad (d)\ \begin{cases} e_1^* = -e_2 \\ e_2^* = -e_1 \end{cases}.$$

It is not difficult to check that if we take the change of basis (a) or (c), then it follows that $\cos t_1 = \cos t_2$. If we take the change of basis (b) or (d), then it follows that $\cos t_1 = -\cos t_2$. It is contradiction.

(3) The proofs of (3) and (4) are similar to the proof of (2). ■

The first part of Theorem 2.1 means that $A^+_{\cos t}$ with positive $\cos t$ (with negative $\cos t$) is isomorphic to $A^-_{\cos t}$ with negative $\cos t$ (with positive $\cos t$).

Case 2. Let $\cos t_2 = -\sin t_2$. In this case, $A^{[t_2]}$ is commutative. According to our assumption $A^{[t_1]}$ is also commutative. It means that $\cos t_1 = -\sin t_1$. Therefore in this case $t_1, t_2 \in \{\frac{3\pi}{4} + \pi k,\ k = 0, 1, 2, \ldots\}$. The possible cases in this case are as follows:

$$(a)\ \begin{cases} t_1 = \dfrac{3\pi}{4} \\ t_2 = \dfrac{3\pi}{4} \end{cases},\quad (b)\ \begin{cases} t_1 = \dfrac{3\pi}{4} \\ t_2 = \dfrac{7\pi}{4} \end{cases},\quad (c)\ \begin{cases} t_1 = \dfrac{7\pi}{4} \\ t_2 = \dfrac{7\pi}{4} \end{cases},\quad (d)\ \begin{cases} t_1 = \dfrac{7\pi}{4} \\ t_2 = \dfrac{3\pi}{4} \end{cases}.$$

For the cases (a) and (c): $\begin{cases} \cos t_1 = \cos t_2 \\ \sin t_1 = \sin t_2 \end{cases}$ and for the cases (b) and (d): $\begin{cases} \cos t_1 = -\cos t_2 \\ \sin t_1 = -\sin t_2 \end{cases}$. In this case from $\cos t_2 = -\sin t_2$ it follows that $\cos t_1 = -\sin t_1$ and one of the solutions of the system (14) is $x_1 = y_2 = 1$, $x_2 = y_1 = 0$ or $x_1 = y_2 = -1$, $x_2 = y_1 = 0$. And we have the following two-dimensional commutative, isomorphic algebras A_{-2} and A_2 with structural constants matrix

$$\mathcal{M}_{-2} = \left(\begin{array}{cc|cc} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{array}\right) \quad \text{and} \quad \mathcal{M}_2 = \left(\begin{array}{cc|cc} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{array}\right)$$

respectively.

From the above results, we obtain the following theorem. In this theorem, we give a time-depending classification of time-homogeneous FA $A^{[t]}$ corresponding to the matrix (9).

Theorem 2.2:

$$A^{[t]} \cong \begin{cases} A_1, & \text{if } t \in \{\pi k : k \in I\} \\ A_0^+, & \text{if } t \in \left\{\dfrac{\pi}{2} + \pi k : k \in I\right\} \\ A_2, & \text{if } t \in \left\{\dfrac{3\pi}{4} + \pi k : k \in I\right\} \\ A^+_{\cos t}, & \text{if } t \in \bigcup\limits_{k \in I} \left(\pi k; \dfrac{\pi}{2} + \pi k\right) \\ A^-_{\cos t}, & \text{if } t \in \bigcup\limits_{k \in I} \left(\left(\dfrac{\pi}{2} + \pi k; \pi + \pi k\right) \setminus \left\{\dfrac{3\pi}{4} + \pi k\right\}\right) \end{cases}$$

where $I = \{0, 1, 2, \ldots\}$ *is set of nonnegative integers and* $\cos t \in (0; 1)$.

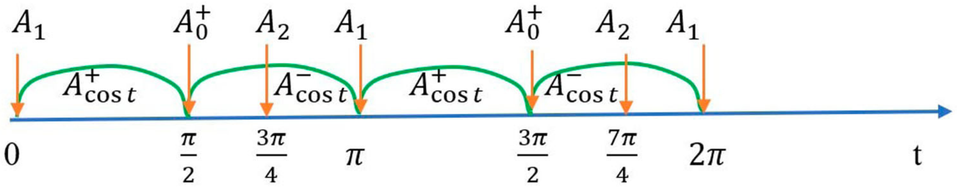

Figure 1. The partition of the time set $\{(0,t) : 0 \leq t\}$ corresponding to the classification of algebras in the FA $A^{[t]}$ with the matrix (9).

We can better see the essence of the theorem using the figure below (Figure 1).

It is known [14] that a finite-dimensional algebra A with a matrix of structural constants $(c_{ijk})_{i,j,k=1}^{m}$ is associative iff

$$\sum_{r=1}^{m} c_{ijr}c_{rkl} = \sum_{r=1}^{m} c_{irl}c_{jkr}, \quad \text{for all } i,j,k,l. \tag{28}$$

Now we check the associativity condition for FA $A^{[t]}$ with the matrix (9). For $m = 2$, the system (27) will be the following form:

$$c_{ij1}c_{1kl} + c_{ij2}c_{2kl} = c_{i1l}c_{jk1} + c_{i2l}c_{jk2} \quad i,j,k,l = 1,2.$$

After some easy calculations we have the following remark.

Remark 2.3: Among the algebras given in Theorem 2.2, only A_1 and A_2 are associative. And note that A_2 is associative and commutative algebra.

Does this FA include evolution algebra? Before answering this question, we recall some notions.

Definition 2.1 ([28]): Let $(E,\cdot)$ be an algebra over a field K. If it admits a basis $e_1, e_2 \cdots$, such that

$$e_i \cdot e_j = 0, \quad \text{if } i \neq j;$$
$$e_i \cdot e_i = \sum_k a_{ik}e_k, \quad \text{for any } i,$$

then this algebra is called an *evolution algebra*. This basis is called natural basis.

Note that an n-dimensional evolution algebra is a non-associative, commutative algebra. In [4], necessary and sufficient conditions for a given commutative algebra to be an evolution algebra were found.

The flow we considered has only one commutative algebra A_2, and it is associative.

Remark 2.4: The FA $A^{[t]}$ corresponding to the matrix (9) does not include any evolution algebra.

3. Comparison with Theorem 1.1

Theorem 3.1: *The algebras mentioned in Theorem 2.2 have the following relations with classes of Theorem 1.1:*

$$A_1 \cong \mathcal{A}_5\left(\frac{1}{2},0\right), \quad A_0^+ \cong \mathcal{A}_8(0,0), \quad A_2 \cong \mathcal{A}_3\left(\frac{1}{2},0,\frac{1}{2}\right),$$
$$A_{\cos t}^+ \cong \mathcal{A}_2\left(\frac{1}{2},0,\frac{-\sin t}{2\cos t}\right), \quad A_{\cos t}^- \cong \mathcal{A}_3\left(\frac{1}{2},0,\frac{\sin t}{2\cos t}\right).$$

Proof: If we write the matrix (6) in the form of the matrix (2), then the matrix of structural constants (9) of the time-homogeneous FA will be the following:

$$\mathcal{M}_e^{[t]} = \begin{pmatrix} \cos t & \cos t & -\sin t & \sin t \\ \sin t & -\sin t & \cos t & \cos t \end{pmatrix}. \tag{29}$$

First we consider the algebra $A_{\cos t}^+$. Now we use the change of basis as in [3]. If we take a change of basis

$$e_1^* = \frac{1}{4\cos t}(e_1 + e_2), \quad e_2^* = \frac{1}{2\sqrt{\sin 2t}}(e_1 - e_2),$$

then the matrix of structural constants of $A_{\cos t}^+$ will be the following:

$$\mathcal{M}_{\cos t, e^*}^+ = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{-\sin t}{2\cos t} & \frac{1}{2} & 0 \end{pmatrix}, \tag{30}$$

where $\sin t, \cos t \in (0;1)$.

It means that $A_{\cos t}^+$ is isomorphic to $\mathcal{A}_2(\frac{1}{2}, 0, \frac{-\sin t}{2\cos t})$.

Similar to the above case we can show that $A_{\cos t}^-$ is isomorphic to $\mathcal{A}_3(\frac{1}{2}, 0, \frac{\sin t}{2\cos t})$. Consequently A_2 is isomorphic to $\mathcal{A}_3(\frac{1}{2}, 0, \frac{1}{2})$.

Now we consider the algebras A_1 and A_0^+. We write their matrices of structural constants similar to (2). Then we get the following matrices respectively

$$\mathcal{M}_{1,e} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{31}$$

and

$$\mathcal{M}_{0,e}^+ = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}. \tag{32}$$

If we take a change $e_1^* = \frac{1}{2}e_1$, $e_2^* = -e_1 + e_2$ for (31), then we get

$$\mathcal{M}_{1,e^*} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

It means that A_1 is isomorphic to $\mathcal{A}_5(\frac{1}{2}, 0)$.

If we take a change $e_1^* = -\frac{1}{2}(e_1 + e_2)$, $e_2^* = \frac{1}{2}(e_1 - e_2)$ for (32), then we get

$$\mathcal{M}^{+}_{0,e^*} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, the algebra A_0^+ is isomorphic to $\mathcal{A}_8(0,0)$. ■

4. Discussion and applications

In this paper, we have fully classified algebras in a flow of two-dimensional algebras defined by cubic matrix (9). Namely, we found condition for two algebras of this flow to be isomorphic at different times. Moreover, we showed that the flow comprises of uncountable pairwise non-isomorphic algebras.

Since isomorphic algebras can be considered essentially the same from an algebraic perspective, as they share all fundamental properties and structures, our Theorem 2.2 is very useful to separate time intervals where corresponding algebras of flow are the same (up to isomorphism). Moreover, the classification depending on time can help in understanding how systems evolve, how symmetries change, or how interactions are structured over time. In particular, this classification is applicable to:

Time evolution of quantum systems: In quantum mechanics, classifying time-dependent algebras helps in studying systems whose Hamiltonians or symmetries evolve over time. This is essential in quantum groups, quantum field theory and quantum computing where time evolution plays a critical role. Different time-dependent algebraic structures correspond to different dynamic symmetries or time-evolving quantum states [7].

Evolving geometric structures: Non-commutative geometry generalizes the concept of space and geometry to non-commutative algebras. Classifying time-dependent algebras helps in modelling physical systems where the underlying ‘space’ is not fixed but evolves with time, such as quantum space–time models [8].

Deformations of algebras: Time-varying representations are connected to the study of deformations of algebraic structures, which arise in areas like deformation quantization and time-varying quantum symmetries [9].

Phase transitions: In materials science and crystallography, the classification of algebras that describe symmetry groups can change during phase transitions (e.g. from solid to liquid or vice versa). Time-dependent algebras model systems where symmetries change as the material undergoes structural or phase changes over time, providing insights into dynamic processes at the microscopic level [11].

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References

[1] Ahmed H, Bekbaev U, Rakhimov I. Complete classification of two-dimensional algebras. AIP Conf Proc. 27 April 2017;1830(1):070016.
[2] Bai R, Liu H, Zhang M. 3-Lie algebras realized by cubic matrices. Chin Ann Math Ser B. 2014;35(2):261–270. doi: 10.1007/s11401-014-0823-2
[3] Bekbaev U. Complete classifications of two-dimensional general, commutative, commutative Jordan, division and evolution real algebras. 2017. arXiv:1705.01237v1.
[4] Bustamante MD, Mellon P, Velasco M. Determining when an algebra is an evolution algebra. Mathematics. 2020;8(8):1349. doi: 10.3390/math8081349
[5] Casas JM, Ladra M, Rozikov UA. A chain of evolution algebras. Linear Algebra Appl. 2011;435(4):852–870. doi: 10.1016/j.laa.2011.02.012
[6] Casas JM, Ladra M, Rozikov UA. Markov processes of cubic stochastic matrices: quadratic stochastic processes. Linear Algebra Appl. 2019;575:273–298. doi: 10.1016/j.laa.2019.04.016
[7] Chaichian M, Demichev AP. Introduction to quantum groups. World Scientific Publishing; 1996.
[8] Connes A. Noncommutative geometry. Academic Press; 1994.
[9] Fox TF. An introduction to algebraic deformation theory. J Pure Appl Algebra. 1993;84:17–41. doi: 10.1016/0022-4049(93)90160-U
[10] Gerstenhaber M. On the deformation of rings and algebras. Ann Math. 1964;79(2):59–103. doi: 10.2307/1970484
[11] Hahn T, editor. International tables for crystallography. Volume A: space group symmetry. Dordrecht: Springer; 2002.
[12] Hartshorne R. Deformation theory. New York (NY): Springer; 2012.
[13] Imomkulov AN, Velasco MV. Chain of three-dimensional evolution algebras. Filomat. 2020;34(10):3175–3190. doi: 10.2298/FIL2010175I
[14] Jacobson N. Lie algebras. New York-London: Interscience Publishers (A Division of John Wiley Sons); 1962. (Interscience tracts in pure and applied mathematics; vol. 10).
[15] Kontsevich M. Deformation quantization of Poisson manifolds. Lett Math Phys. 2003;66: 157–216. doi: 10.1023/B:MATH.0000027508.00421.bf
[16] Ladra M, Rozikov UA. Flow of finite-dimensional algebras. J Algebra. 2017;470:263–288. doi: 10.1016/j.jalgebra.2016.09.011
[17] Ladra M, Rozikov UA. Construction of flows of finite-dimensional algebras. J Algebra. 2017;492:475–489. doi: 10.1016/j.jalgebra.2017.09.015
[18] Ladra M, Rozikov UA. Algebras of cubic matrices. Linear Multilinear Algebra. 2017;65(7): 1316–1328. doi: 10.1080/03081087.2016.1234581
[19] Lilly JM. Kinematics of a fluid ellipse in a linear flow. Fluids. 2018;3(1):16. doi: 10.3390/fluids3010016
[20] Maksimov VM. Cubic stochastic matrices and their probability interpretations. Theory Probab Appl. 1996;41(1):55–69. doi: 10.1137/TPRBAU000041000001000055000001
[21] Murodov SN. Classification dynamics of two-dimensional chains of evolution algebras. Int J Math. 2014;25(2):1450012. doi: 10.1142/S0129167X14500128
[22] Narkuziyev BA, Rozikov UA. Classification in chains of three-dimensional real evolution algebras. Linear Multilinear Algebra. 2023;71(2):265–300. doi: 10.1080/03081087.2022.2025754.

[23] Ogden RW. Non-linear elastic deformations. New York: Dover Publications, Inc.; 1984.

[24] Omirov BA, Rozikov UA, Tulenbayev KM. On real chains of evolution algebras. Linear Multilinear Algebra. 2015;63(3):586–600. doi: 10.1080/03081087.2014.889131

[25] Paniello I. Marginal distributions of genetic coalgebras. J Math Biol. 2014;68(5):1071–1087. doi: 10.1007/s00285-013-0663-9

[26] Rozikov UA. Population dynamics: algebraic and probabilistic approach. Singapore: World Scientific Publishing; 2020. 460 pp.

[27] Rozikov UA, Murodov SN. Dynamics of two-dimensional evolution algebras. Lobachevskii J Math. 2013;34(4):344–358. doi: 10.1134/S199508021304015X

[28] Tian JP. Evolution algebras and their applications. Berlin: Springer-Verlag; 2008. (Lecture notes in mathematics; 1921).