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Maximal Operators Associated with Singular Surfaces

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Abstract. Maximal operators associated with a class of singular parametrized surfaces in \mathbb{R}^3 are analyzed in the paper. Boundedness of such operators in Lebesgue L^p space for $p > 2$ is shown. It is also proved that at least one of the principal curvatures does not vanish at each regular point of these surfaces.

Keywords: Maximal operator, averaging operator, fractional power series, singular surface, principal curvatures.

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1. Introduction and preliminaries

Let S be a hypersurface in \mathbb{R}^{n+1} with a surface measure dS . Let us denote the "average" of f on \mathbb{R}^{n+1} along the t -dilate of the hypersurface S as

$$\mathcal{A}_t f(y) := \int_S f(y - tx) \psi(x) dS(x), \quad t > 0,$$

where ψ is a non-negative infinitely smooth function with compact support, i.e., $0 \leq \psi \in C_0^\infty(\mathbb{R}^{n+1})$ and $f \in C_0^\infty(\mathbb{R}^{n+1})$.

The associated maximal operator is defined by the following relation

$$\mathcal{M}f(y) := \sup_{t>0} | \mathcal{A}_t f(y) |. \quad (1)$$

The maximal operator (1) is said to be bounded on $L^p := L^p(\mathbb{R}^{n+1})$ if there exists $C > 0$ such that for any function $f \in C_0^\infty(\mathbb{R}^{n+1})$ the inequality

$$\| \mathcal{M}f \|_{L^p} \leq C \| f \|_{L^p}$$

holds true, where $\| \cdot \|_{L^p}$ is the natural norm of space L^p .

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The prototype of such kind of maximal operators is Stein's maximal function (in which S is the Euclidean unit sphere centred at the origin). It is known [1] that spherical maximal operator (1) is bounded on $L^p(\mathbb{R}^n)$ for $p > \frac{n}{n-1}$, $n \geq 3$ and it is not bounded on $L^p(\mathbb{R}^n)$ whenever $p \leq \frac{n}{n-1}$. The two dimensional case of Stein's result was considered by Bourgain [2].

Then Greenleaf proved that maximal operator (1) is bounded on $L^p(\mathbb{R}^{n+1})$ for $n \geq 2$ and $p > (n+1)/n$ when hypersurface has everywhere non-vanishing Gaussian curvature [3]. Moreover, Greenleaf showed that if hypersurface has at least $k(k \geq 2)$ non-vanishing principal curvatures then maximal operator is bounded on $L^p(\mathbb{R}^{n+1})$ ($n \geq 2$) for all $p > (k+1)/k$. Later similar result for more difficult case $k = 1$ was obtained by Sogge [4].

Also, maximal operators (1) have been investigated [5–9]. Boundedness of maximal operators related to singular surfaces in three and multidimensional Euclidean spaces was studied [10–16].

2. Statement of the problem

Definition of the fractional power series is given in [17].

Definition 1. Let $V \subseteq \mathbb{R}_+^n$ be an open connected set such that $0 \in \bar{V}$, f is called a fractional power series in set V if there is an open set $W \subseteq \mathbb{R}^n$ containing \bar{V} , a natural number N and a real analytic function g in $\Phi_N^{-1}(W)$ such that the identity $f = g \circ \Phi_{1/N}$ holds in set V , where $\Phi_N : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a map given by the formula $\Phi_N(x) = (x_1^N, x_2^N, \dots, x_n^N)$.

Let us consider a class of singular parametrized surfaces in \mathbb{R}^3 defined by

$$\begin{aligned} x_1(u_1, u_2) &= r_1 + u_1^{a_1} u_2^{a_2} g_1(u_1, u_2), \quad x_2(u_1, u_2) = r_2 + u_1^{b_1} u_2^{b_2} g_2(u_1, u_2), \\ x_3(u_1, u_2) &= r_3 + u_1^{c_1} u_2^{c_2} g_3(u_1, u_2), \end{aligned} \quad (2)$$

where r_1, r_2, r_3 are any real numbers, $a_1, a_2, b_1, b_2, c_1, c_2$ are non-negative rational numbers, $u_1 \geq 0, u_2 \geq 0$ and $\{g_k(u_1, u_2)\}_{k=1}^3$ are fractional power series.

For further statements the following designations are introduced

$$B_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad B_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

L^p – boundedness of the maximal operator (1) was studied in a small neighborhood of the point $(r_1, r_2, r_3) \neq (0, 0, 0)$ [10–12, 15, 16], when $p > 2$ and at least two of the numbers B_1, B_2, B_3 are non-zero. L^p ($p > 2$) – boundedness of the maximal operators related to singular hypersurfaces in the multidimensional Euclidean spaces was studied [13, 14].

In this paper, the case $r_1 = r_2 = r_3 = 0$ in (2) is investigated. More precisely, the problem of L^p ($p > 2$) – boundedness of the maximal operator (1) along the parametrized surfaces of the form

$$x_1(u_1, u_2) = u_1^{a_1} u_2^{a_2} g_1(u_1, u_2), \quad x_2(u_1, u_2) = u_1^{b_1} u_2^{b_2} g_2(u_1, u_2), \quad x_3(u_1, u_2) = u_1^{c_1} u_2^{c_2} g_3(u_1, u_2) \quad (3)$$

is studied. These surfaces satisfy the following condition.

Remark 1. If at least one of the numbers B_1, B_2, B_3 is non-zero then points of surface (3) lying in a sufficiently small neighbourhood of the origin in \mathbb{R}_+^3 outside the coordinate planes are regular (nonsingular). Points of surface (3) lying on the coordinate planes may be singular (see definition 2 in [10]).

The averaging operator associated with surface (3) is defined as follows

$$\mathcal{A}_t^g f(y) = \int_{\mathbb{R}_+^2} f\left(y_1 - tu_1^{a_1} u_2^{a_2} g_1(u_1, u_2), y_2 - tu_1^{b_1} u_2^{b_2} g_2(u_1, u_2), y_3 - tu_1^{c_1} u_2^{c_2} g_3(u_1, u_2)\right) \times \\ \times \psi(u_1, u_2) du_1 du_2.$$

Then corresponding maximal operator is defined by

$$\mathcal{M}^g f(y) := \sup_{t>0} |\mathcal{A}_t^g f(y)|, \quad y \in \mathbb{R}^3.$$

For the case of smooth surfaces it corresponds to a surface integral.

3. Auxiliary assertions

Let us state some necessary facts known from geometry which are used to prove main result. As usual, the first and second fundamental forms of surfaces (3) have the following forms

$$G(u, du) = g_{11}(u_1, u_2) du_1^2 + 2g_{12}(u_1, u_2) du_1 du_2 + g_{22}(u_1, u_2) du_2^2, \\ L(u, du) = l_{11}(u_1, u_2) du_1^2 + 2l_{12}(u_1, u_2) du_1 du_2 + l_{22}(u_1, u_2) du_2^2,$$

respectively. Coefficients of the fundamental forms are calculated by the following formulas

$$g_{ij} = g_{ij}(u_1, u_2) = (\bar{r}_i \bar{r}_j), \quad l_{ij} = l_{ij}(u_1, u_2) = (\bar{r}_i, \bar{m}), \quad (4)$$

where $\bar{r}_i = \frac{\partial \bar{r}}{\partial u_i}$, $\bar{r}_{ij} = \frac{\partial^2 \bar{r}}{\partial u_i \partial u_j}$, $i, j = 1, 2$, \bar{m} is the unit normal vector at each regular point of surfaces (3).

Next let us use the change of variables

$$u_1 = u_1(z_1, z_2), \quad u_2 = u_2(z_1, z_2) \quad (5)$$

in (3), where $u_1(z_1, z_2), u_2(z_1, z_2)$ are differentiable functions in a neighbourhood of zero in \mathbb{R}^2 . Jacobi matrix of transformation (5) is denoted by J . Then the following lemma is valid.

Lemma 1. *If $\det J \neq 0$ then equalities $|\tilde{L} - \lambda \tilde{G}| = 0$ and $|L - \lambda G| = 0$ are equivalent equations, where \tilde{G} and \tilde{L} are matrices of the first and second fundamental forms of surfaces (3) after the change of variables (5).*

Proof. Indeed, applying the change of coordinates (5), matrices g_{ij} are transformed by the following rules

$$\tilde{g}_{ij}(z_1, z_2) = \sum_{k,l} g_{kl}(u_1, u_2) \frac{\partial u_k}{\partial z_i} \frac{\partial u_l}{\partial z_j}, \quad k, l = 1, 2$$

or in the matrix form $\tilde{G} = J^T G J$, and, analogously, $\tilde{L} = J^T L J$ (see [18], chapter 3, Sec. 16).

Obviously, principal curvatures of surfaces (3) are defined as solutions of the equation $|L - \lambda G| = 0$. It is easy to show that equality

$$|\tilde{L} - \lambda \tilde{G}| = (\det J)^2 |L - \lambda G|$$

holds true for surfaces (3). Hence, this equation indicates that lemma is correct. \square

The proof of main result of this paper is based on the following result due to Sogge [4].

Theorem 1. *Let S be a smooth hypersurface in \mathbb{R}^n , $n \geq 2$, with the property that at each $x \in S$ at least one principal curvature is non-zero. With $\mathcal{M}f$ as in (1) above then $\mathcal{M}f$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 2$.*

4. Main result

The main result of the paper is the following theorem.

Theorem 2. *Let $\{g_k(u_1, u_2)\}_{k=1}^3$ be fractional power series which defined in a small neighbourhood of the origin of coordinate system of \mathbb{R}^2 and satisfy the following conditions: $g_k(0, 0) \neq 0$. Suppose that at least one of the following conditions is satisfied:*

1. $B_1 \neq 0$ and either $B_2B_3 \neq 0$, or $B_2(B_2 + B_1) \neq 0$, or $B_3(B_3 - B_1) \neq 0$;
2. $B_3 \neq 0$ and either $B_2B_1 \neq 0$, or $B_2(B_2 - B_3) \neq 0$, or $B_1(B_1 - B_3) \neq 0$;
3. $B_2 \neq 0$ and either $B_1B_3 \neq 0$, or $B_3(B_3 - B_2) \neq 0$, or $B_1(B_1 + B_2) \neq 0$.

Then there exists a small neighbourhood U of the origin in \mathbb{R}^3 , such that for any function $\psi \in C_0^\infty(U)$ maximal operator $\mathcal{M}^g f$ is bounded on $L^p(\mathbb{R}^3)$ for $p > 2$.

Proof. Suppose that condition 1 is satisfied. Let us study the boundedness of maximal operator $\mathcal{M}^g f$ at nonsingular points of surface (3) (see Remark 1).

Using the change of variables

$$u_1 = 2^{-j_1} v_1, \quad u_2 = 2^{-j_2} v_2$$

in (3), one can obtain

$$\begin{aligned} \tilde{x}_1(v_1, v_2) &= 2^{-(j_1 a_1 + j_2 a_2)} v_1^{a_1} v_2^{a_2} g_1(2^{-j_1} v_1, 2^{-j_2} v_2), \\ \tilde{x}_2(v_1, v_2) &= 2^{-(j_1 b_1 + j_2 b_2)} v_1^{b_1} v_2^{b_2} g_2(2^{-j_1} v_1, 2^{-j_2} v_2), \\ \tilde{x}_3(v_1, v_2) &= 2^{-(j_1 c_1 + j_2 c_2)} v_1^{c_1} v_2^{c_2} g_3(2^{-j_1} v_1, 2^{-j_2} v_2), \end{aligned} \tag{6}$$

where $0, 5 \leq v_1 \leq 2$, $0, 5 \leq v_2 \leq 2$, $j_1, j_2 \geq j_0$, j_0 is a large number such that implies from the smallness of the support of ψ .

Next, let us introduce the change of variables

$$\begin{cases} w_1 = v_1^{a_1} v_2^{a_2} g_1(2^{-j_1} v_1, 2^{-j_2} v_2) \\ w_2 = v_1^{b_1} v_2^{b_2} g_2(2^{-j_1} v_1, 2^{-j_2} v_2), \end{cases} \tag{7}$$

and assume that $g_1(0, 0) = g_2(0, 0) = 1$. If $g_1(0, 0) < 0$, $g_2(0, 0) < 0$, then one can obtain $-g_1(u_1, u_2)$, $-g_2(u_1, u_2)$ instead of $g_1(u_1, u_2)$, $g_2(u_1, u_2)$.

Now, calculating Jacobian of coordinate change (7), one can find

$$\det J_1 = \begin{vmatrix} \frac{\partial w_1}{\partial v_1} & \frac{\partial w_2}{\partial v_1} \\ \frac{\partial w_1}{\partial v_2} & \frac{\partial w_2}{\partial v_2} \end{vmatrix} = v_1^{a_1 + b_1 - 1} v_2^{a_2 + b_2 - 1} \left(B_1 g_1(2^{-j_1} v_1, 2^{-j_2} v_2) g_2(2^{-j_1} v_1, 2^{-j_2} v_2) + h(v_1, v_2) \right),$$

where

$$\begin{aligned} h(v_1, v_2) &= v_1 h_1(v_1, v_2) + v_2 h_2(v_1, v_2) + v_1 v_2 h_3(v_1, v_2), \\ h_1(v_1, v_2) &= b_2 g_2(2^{-j_1} v_1, 2^{-j_2} v_2) \frac{\partial g_1(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_1} - a_2 g_1(2^{-j_1} v_1, 2^{-j_2} v_2) \frac{\partial g_2(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_1}, \\ h_2(v_1, v_2) &= a_1 g_1(2^{-j_1} v_1, 2^{-j_2} v_2) \frac{\partial g_2(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_2} - b_1 g_2(2^{-j_1} v_1, 2^{-j_2} v_2) \frac{\partial g_1(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_2}, \\ h_3(v_1, v_2) &= \frac{\partial g_1(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_1} \frac{\partial g_2(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_2} - \frac{\partial g_1(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_2} \frac{\partial g_2(2^{-j_1} v_1, 2^{-j_2} v_2)}{\partial v_1} \end{aligned}$$

– fractional power series. It is not difficult to see that $\det J_1 \neq 0$ at the point (v_1, v_2) .

Let us consider the system

$$\begin{cases} w_1 = v_1^{a_1} v_2^{a_2} \\ w_2 = v_1^{b_1} v_2^{b_2}, \end{cases}$$

in the first quadrant \mathbb{R}_+^2 and obtain

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}}. \end{cases} \quad (8)$$

In particular, relation (8) is in the set

$$\{(w_1, w_2) \in \mathbb{R}_+^2 : 2^{-(a_1+a_2)} \leq w_1 \leq 2^{a_1+a_2}, 2^{-(b_1+b_2)} \leq w_2 \leq 2^{b_1+b_2}\}$$

Further, let us also introduce the change of variables

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1 \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2 \end{cases} \quad (9)$$

where \hat{g}_1, \hat{g}_2 are new variables and suppose that $\hat{g}_1 \sim 1, \hat{g}_2 \sim 1$. As a result, system (7) implies

$$\begin{aligned} (\hat{g}_1)^{a_1} (\hat{g}_2)^{a_2} g_1 \left(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1, 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2 \right) &= 1 \\ (\hat{g}_1)^{b_1} (\hat{g}_2)^{b_2} g_2 \left(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \hat{g}_1, 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \hat{g}_2 \right) &= 1. \end{aligned} \quad (10)$$

Analogously, one can show that Jacobian of system (10) is non-zero in a small neighbourhood of the point $(0, 0, 1, 1)$. In view of the implicit function theorem this system has smooth solutions with respect to \hat{g}_1, \hat{g}_2 of the form

$$\tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) = 1 + 2^{-j_1} \tilde{h}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) + 2^{-j_2} \tilde{h}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2),$$

$$\tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2) = 1 + 2^{-j_1} \tilde{\rho}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) + 2^{-j_2} \tilde{\rho}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2)$$

in a sufficiently small neighborhood of the point $(0, 0, 1, 1)$. Here $\tilde{h}_1, \tilde{h}_2, \tilde{\rho}_1, \tilde{\rho}_2$ are smooth functions. It is assumed that $\tilde{g}_1(0, 0, 1, 1) = 1, \tilde{g}_2(0, 0, 1, 1) = 1$.

Then taking into account (9), one can obtain

$$\begin{cases} v_1 = w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2) \\ v_2 = w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2). \end{cases} \quad (11)$$

By applying relations (7) and (11) to equations (6), one can find

$$\hat{x}_1(w_1, w_2) = 2^{-(j_1 a_1 + j_2 a_2)} w_1, \hat{x}_2(w_1, w_2) = 2^{-(j_1 b_1 + j_2 b_2)} w_2,$$

$$\hat{x}_3(w_1, w_2) = 2^{-(j_1 c_1 + j_2 c_2)} \alpha(w_1, w_2), \quad (12)$$

where $\alpha(w_1, w_2) = w_1^{-\frac{B_2}{B_1}} w_2^{\frac{B_3}{B_1}} g(w_1, w_2)$,

$$g(w_1, w_2) = (\tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2))^{c_1} (\tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2))^{c_2} \times \\ \times g_3\left(2^{-j_1} w_1^{\frac{b_2}{B_1}} w_2^{\frac{-a_2}{B_1}} \tilde{g}_1(2^{-j_1}, 2^{-j_2}, w_1, w_2), 2^{-j_2} w_1^{\frac{-b_1}{B_1}} w_2^{\frac{a_1}{B_1}} \tilde{g}_2(2^{-j_1}, 2^{-j_2}, w_1, w_2)\right).$$

It is well-known that second fundamental form of the surface given by parametric equations (12) has the following form

$$L = L_{11}dw_1^2 + 2L_{12}dw_1dw_2 + L_{22}dw_2^2.$$

Using (4), one can find

$$L_{11} = (\bar{r}_{11}, \bar{n}), \quad L_{12} = (\bar{r}_{12}, \bar{n}), \quad L_{22} = (\bar{r}_{22}, \bar{n}), \quad (13)$$

$$\bar{r}_{11} = \frac{\partial^2 \bar{r}}{\partial w_1^2}, \quad \bar{r}_{12} = \frac{\partial^2 \bar{r}}{\partial w_1 \partial w_2}, \quad \bar{r}_{22} = \frac{\partial^2 \bar{r}}{\partial w_2^2},$$

$\bar{n} = \bar{N} \cdot |\bar{N}|^{-1}$ is unit normal vector. A normal vector \bar{N} in any point of surface (12) defined by

$$\bar{N} = \begin{vmatrix} i & j & k \\ \frac{\partial \hat{x}_1}{\partial w_1} & \frac{\partial \hat{x}_2}{\partial w_1} & \frac{\partial \hat{x}_3}{\partial w_1} \\ \frac{\partial \hat{x}_1}{\partial w_2} & \frac{\partial \hat{x}_2}{\partial w_2} & \frac{\partial \hat{x}_3}{\partial w_2} \end{vmatrix}.$$

Now calculating coefficients L_{11} , L_{22} and L_{12} , one can obtain

$$L_{11} = \frac{\partial^2 \alpha}{\partial w_1^2} = C w_1^{-\frac{B_2}{B_1}-2} w_2^{\frac{B_3}{B_1}} \left(B_2(B_2 + B_1)g(w_1, w_2) - B_2 B_1 w_1 \frac{\partial g(w_1, w_2)}{\partial w_1} + B_1^2 w_1^2 \frac{\partial^2 g(w_1, w_2)}{\partial w_1^2} \right),$$

$$L_{22} = \frac{\partial^2 \alpha}{\partial w_2^2} = C w_1^{-\frac{B_2}{B_1}} w_2^{\frac{B_3}{B_1}-2} \left(B_3(B_3 - B_1)g(w_1, w_2) + B_3 B_1 w_2 \frac{\partial g(w_1, w_2)}{\partial w_2} + B_1^2 w_2^2 \frac{\partial^2 g(w_1, w_2)}{\partial w_2^2} \right),$$

$$L_{12} = \frac{\partial^2 \alpha}{\partial w_1 \partial w_2} = -C w_1^{-\frac{B_2}{B_1}-1} w_2^{\frac{B_3}{B_1}-1} \left(B_2 B_3 g(w_1, w_2) - B_3 B_1 w_1 \frac{\partial g(w_1, w_2)}{\partial w_1} + B_2 B_1 w_2 \frac{\partial g(w_1, w_2)}{\partial w_2} - B_1^2 w_1 w_2 \frac{\partial^2 g(w_1, w_2)}{\partial w_1 \partial w_2} \right),$$

where $C = 2^{-(j_1 c_1 + j_2 c_2)} \frac{|\bar{N}|^{-1}}{B_1^2}$.

It follows from condition 1 that at least one of numbers $B_2 B_3$, $B_2(B_2 + B_1)$ and $B_3(B_3 - B_1)$ is not equal to zero. Therefore, at least one of coefficients L_{11} , L_{12} and L_{22} is non-zero at each regular point of surfaces (12) in a small neighbourhood of the origin for sufficiently large j_0 .

Consequently, singular surfaces (12) as well as singular surfaces given by (3) have at least one non vanishing principal curvature at each regular point of these surfaces.

Thus, making similar arguments under conditions 2 or 3, one can show that in a small neighbourhood of the origin of \mathbb{R}_+^3 at least one of the principal curvatures of singular surfaces (3) is non-zero. Therefore, in view of Theorem 1 maximal operator $\mathcal{M}^g f$ is bounded on $L^p(\mathbb{R}^3)$ for $p > 2$. This completes the proof of Theorem 2.

Let us now consider some corollaries in connection with Theorem 2.

Corollary 1. Let $\{g_k(u_1, u_2)\}_{k=1}^3$ be real analytic functions defined in a small neighbourhood of the origin of coordinate system of \mathbb{R}^2 and satisfy the following conditions: $g_k(0, 0) \neq 0$. Then under assumptions of Theorem 2 its assertions are true.

Corollary 2. If conditions 1-3 of Theorem 2 are replaced with the relations

1. $B_1 \neq 0, A_1^{-1}\bar{c} \neq (1, 0)^T, A_1^{-1}\bar{c} \neq (0, 1)^T, A_1^{-1}\bar{c} \neq (0, 0)^T$;
2. $B_2 \neq 0, A_2^{-1}\bar{a} \neq (1, 0)^T, A_2^{-1}\bar{a} \neq (0, 1)^T, A_2^{-1}\bar{a} \neq (0, 0)^T$;
3. $B_3 \neq 0, A_3^{-1}\bar{b} \neq (1, 0)^T, A_3^{-1}\bar{b} \neq (0, 1)^T, A_3^{-1}\bar{b} \neq (0, 0)^T$,

respectively, and other conditions are satisfied then assertions of Theorem 2 hold true. Here A_1, A_2, A_3 are matrices of B_1, B_2, B_3 , respectively, and $\bar{a} = (a_1, a_2)^T, \bar{b} = (b_1, b_2)^T, \bar{c} = (c_1, c_2)^T$.

References

- [1] E.M.Stein, Maximal functions. Spherical means, *Proc.Nat. Acad. Sci. U.S.A.* **73**(1976), no. 7, 2174–2175.
- [2] J.Bourgain, Averages in the plane convex curves and maximal operators, *J.Anal. Math.*, **47**(1986), 69–85.
- [3] A.Greenleaf, Principal curvature and harmonic analysis, *Indiana Univ. Math. J.*, **30**(1981), 519–537.
- [4] C.D.Sogge, Maximal operators associated to hypersurfaces with one nonvanishing principal curvature, *Fourier analysis and partial differential equations, Stud. Adv. Math.*, (1995), 317–323.
- [5] C.D.Sogge, E.M.Stein, Averages of functions over hypersurfaces in \mathbb{R}^n , *Inventiones mathematicae.*, **82**(1985), 543–556.
- [6] I.A.Ikromov, M.Kempe, D.Müller, Damped oscillatory integrals and boundedness of maximal operators associated to mixed homogeneous hypersurfaces, *Duke Math. J.*, **126**(2005), 471–490. DOI: 10.1215/S0012-7094-04-12632-6
- [7] I.A.Ikromov, M.Kempe, D.Müller, Estimates for maximal functions associated to hypersurfaces in \mathbb{R}^3 and related problems of harmonic analysis, *Acta. Math.*, **204**(2010), 151–271. DOI: 10.1007/s11511-010-0047-6
- [8] S.Buschenhenke, S.Dendrinos, I.A.Ikromov, D. Müller, Estimates for maximal functions associated to hypersurfaces in \mathbb{R}^3 with height $h < 2$: Part I, *Trans. Amer. Math. Soc.* **372**(2019), no. 2, 1363–1406.
- [9] I.A.Ikromov, S.E.Usmanov, On boundedness of maximal operators associated with hypersurfaces, *J. Math. Sci.*, **264**(2022), 715–745.
- [10] S.E.Usmanov, The Boundedness of Maximal Operators Associated with Singular Surfaces, *Russ. Math.*, **65**(2021), 73–83. DOI: 10.3103/S1066369X21060086
- [11] S.E.Usmanov, On the Boundedness Problem of Maximal Operators, *Russ. Math.*, **66**(2022), 74–83. DOI: 10.3103/S1066369X22040077
- [12] S.E.Usmanov, On Maximal Operators Associated with a Family of Singular Surfaces, *J. Sib. Fed. Univ. Math. Phys.*, **16**(2023), no. 2, 265–274. EDN: UBLKLU

- [13] S.E.Usmanov, On the Boundedness of the Maximal Operators Associated with Singular Hypersurfaces. *Math. Notes.*, **114**(2023), no. 1-2, 108–116. DOI: 10.1134/S0001434623070118
- [14] S.E.Usmanov, On Maximal Operators Associated with Singular Hypersurfaces, *Russ. Math.*, **68**(2024), 61–68. DOI: 10.3103/S1066369X24700051
- [15] S.E.Usmanov, On the Boundedness of Maximal Operators Associated with Singular Surfaces, *J. Sib. Fed. Univ. Math. Phys.*, **17**(2024), no. 4, 455–463. EDN: AVGDOC
- [16] S.E.Usmanov, On Estimates for Maximal Operators Associated with Singular Surfaces, *Lobachevskii J. Math.*, **45**(2024), no. 2, 875–882. DOI: 10.1134/S1995080224600146
- [17] T.Collins, A.Greenleaf, M.Pramanik, A multi-dimensional resolution of singularities with applications to analysis, *Amer. J. of Math.*, **135**(2013), 1179–1252.
- [18] B.A.Dubrovin, S.P.Novikov, A.T.Fomenko, Contemporary geometry, Published in "Nauka", Moscow, 1979 (in Russian).

Максимальные операторы, связанные с сингулярными поверхностями

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Аннотация. Статья анализирует максимальные операторы, ассоциированные с классом сингулярных параметризованных поверхностей в \mathbb{R}^3 , показывая ограниченность таких операторов в пространстве Лебега L^p при $p > 2$. Также доказано, что хотя бы одна из главных кривизн не обращается в нуль в каждой регулярной точке этих поверхностей.

Ключевые слова: максимальный оператор, оператор усреднения, дробно-степенной ряд, сингулярная поверхность, главные кривизны.